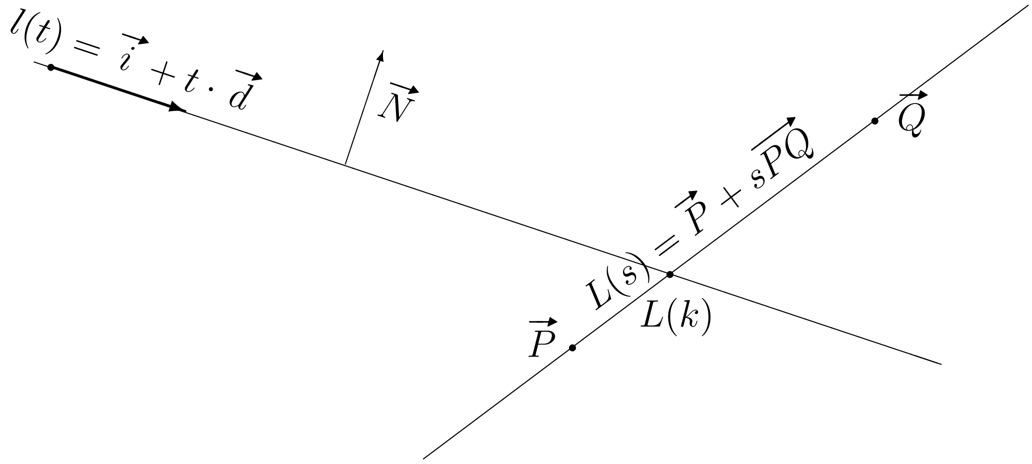
Theorem. Any line l divides the plane into two parts such that a line L connecting a point P on one side of l to Q on the other side of l must intersect l.



Proof. Take some line $l(t) = \vec{i} + t \cdot \vec{d}$. Let $\vec{N}$ be some vector such that $\vec{N} \nparallel \vec{d}$*. Then, $\vec{N}$ and $\vec{d}$ form a basis.

We then define $\vec{P}$ and $\vec{Q}$ in the form $\vec{i} + a\vec{d} + b\vec{N}$; take some $\vec{P} = \vec{i} + P_d\vec{d} + P_N\vec{N}$ and $\vec{Q} = \vec{i} + Q_d\vec{d} + Q_N\vec{N}$. We'd like for one to be "on one side of l" and the other to be "on the other side", so we assert that $P_N < 0$ and $Q_N > 0$.

Note, then, that:

$$\begin{aligned}
\overrightarrow{PQ} &= \langle Q_0 - P_0, Q_1 - P_1 \rangle \\
&= \langle (i_0 + Q_d d_0 + Q_N N_0) - (i_0 + P_d d_0 + P_N N_0), \\
&\quad (i_1 + Q_d d_1 + Q_N N_1) - (i_1 + P_d d_1 + P_N N_1) \rangle \\
&= \quad \langle i_0 - i_0, i_1 - i_1 \rangle \\
&\quad + \langle Q_d d_0 - P_d d_0, Q_d d_1 - P_d d_1 \rangle \\
&\quad + \langle Q_N N_0 - P_N N_0, Q_N N_1 - P_N N_1 \rangle \\
&= (Q_d - P_d)\vec{d} + (Q_N - P_N)\vec{N}
\end{aligned}$$

Let L be the line $\overrightarrow{PQ}$; define $L(s) = \vec{P} + s\overrightarrow{PQ}$. Also let $N(\vec{x})$ denote the $\vec{N}$ component of $\vec{x}$ when expressed as $\vec{i} + a\vec{d} + b\vec{N}$. Then for some s,

$$\begin{aligned}
N(L(s)) &= N(\vec{P} + s\overrightarrow{PQ}) \\
&= N((\vec{i} + P_d\vec{d} + P_N\vec{N}) + (s[(Q_N - P_N)\vec{N} + (Q_d - P_d)\vec{d}])) \\
&= N(\vec{i} + P_d\vec{d} + P_N\vec{N} + s(Q_N - P_N)\vec{N} + s(Q_d - P_d)\vec{d}) \\
&= N(\qquad P_d\vec{d} + P_N\vec{N} + s(Q_N - P_N)\vec{N} + s(Q_d - P_d)\vec{d}) \text{ since } \vec{i} \text{ is on } l \\
&= P_N + s(Q_N - P_N)
\end{aligned}$$

Note that $N(\vec{P}) = P_N < 0$ and $N(\vec{Q}) = Q_N > 0$. Note as well that $L(0) = \vec{P}$ and $L(1) = \vec{Q}$. Thus, $(N \circ L)(0) < 0$ and $(N \circ L)(1) > 0$. Since $N \circ L$ is continuous, then by the Intermediate Value Theorem, $\exists k \mid (N \circ L)(k) = 0$.

Consider $L(k)$ when expressed as $\vec{i} + a\vec{d} + b\vec{N}$. We know that $N(L(k)) = 0$, so $b = 0$ and $L(k) = \vec{i} + a\vec{d} = l(a)$. Thus, $\exists k, a \mid L(k) = l(a)$, and therefore L and l intersect. □

*$\vec{N}$ certainly exists; let $\vec{N} = \langle -d_0, -d_1 \rangle$.